1. Let G be the set of isometries $f: S \longrightarrow S$

Let $f_1, f_2, f_3 \in G$, $p \in S$, $v, w \in T_p S$

(I) Consider $f_1 \circ f_2$, $f_1 \circ f_2$ is ditteomorphism from S to S

$$\langle d(f_1 \circ f_2)_p(v), d(f_1 \circ f_2)_p(w) \rangle$$

$$= \langle df_{1_{f(p)}}(df_{2_p}(v)), df_{1_{f(p)}}(df_{2_p}(w)) \rangle$$

$$= \langle df_{2_p}(v), df_{2_p}(w) \rangle \quad (\text{since } f_1 \text{ is isometry})$$

$$= \langle v, w \rangle$$

Therefore, $f_1 \circ f_2$ is isometry

Hence G is closed under composition (i.e. if $f_1, f_2 \in G$, $f_1 \circ f_2 \in G$)

(II) $f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3$

(III) Consider f_1^{-1}, f_1^{-1} is ditteomorphism from S to S

$$\langle df^{-1}_{1_p}(v), df^{-1}_{1_p}(w) \rangle = \langle df_{f^{-1}(p)}(df^{-1}_p(v)), df_{f^{-1}(p)}(df^{-1}_p(w)) \rangle$$

$$= \langle v, w \rangle$$

f_1^{-1} is also in G

(IV) The identity map is isometry. Identity map: $Id(x) = x$

So G with composition $\circ$ form a group

The isometry group of S^2 is $O(3)$

2. Consider $X(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$ $u \in \mathbb{R}$, $v \in (0, 2\pi)$

is parametrization of catenoid

$\langle X_u, X_u \rangle = \cosh^2 u$, $\langle X_u, X_v \rangle = 0$, $\langle X_v, X_v \rangle = \cosh^2 u$

Consider $Y(u,v) = (\sinh u \cos v, \sinh u \sin v, v)$ $\quad u,v \in \mathbb{R}$

is parametrization of helicoid

$\langle Y_u, Y_u \rangle = \cosh^2 u$, $\langle Y_u, Y_v \rangle = 0$, $\langle Y_v, Y_v \rangle = \cosh^2 u$

By HW8 question 3, $f = Y \circ X^{-1} : S_1 \to S_2$ is local isometry

No isometry between helicoid and catenoid

3. $\alpha(t) : (-\varepsilon, \varepsilon) \longrightarrow S$ be curve on $S \subset \mathbb{R}^3$

$X(t), Y(t)$: tangential vector fields along α

$$\begin{aligned}\frac{d}{dt}\langle X(t), Y(t)\rangle &= \left\langle \frac{d}{dt}X(t), Y(t)\right\rangle + \left\langle X(t), \frac{d}{dt}Y(t)\right\rangle \\ &= \left\langle \left(\frac{d}{dt}X(t)\right)^T + \left(\frac{d}{dt}X(t)\right)^\perp, Y(t)\right\rangle + \left\langle X(t), \left(\frac{d}{dt}Y(t)\right)^T + \left(\frac{d}{dt}Y(t)\right)^\perp\right\rangle \\ &= \left\langle \left(\frac{d}{dt}X(t)\right)^T, Y(t)\right\rangle + \left\langle X(t), \left(\frac{d}{dt}Y(t)\right)^T\right\rangle \\ &= \langle \nabla_{\alpha'(t)}X(t), Y(t)\rangle + \langle X(t), \nabla_{\alpha'(t)}Y(t)\rangle\end{aligned}$$

v^T: tangential component of v

$v^\perp$: normal component of v

$\left(\left\langle \left(\frac{d}{dt}X(t)\right)^\perp, Y(t)\right\rangle = 0 = \left\langle X(t), \left(\frac{d}{dt}Y(t)\right)^\perp\right\rangle \right)$

If X, Y are parallel vector field along α, (i.e. $\nabla_{\alpha'(t)}X(t) = 0 = \nabla_{\alpha'(t)}Y(t)$)

we claim angle between X and Y is constant.

We need to show $\|X\|$, $\|Y\|$, $\langle X, Y\rangle$ are constant.

$\frac{d}{dt}\langle X(t), X(t)\rangle = 2\langle \nabla_{\alpha'(t)}X(t), X(t)\rangle = 0$

$\frac{d}{dt}\langle Y(t), Y(t)\rangle = 2\langle \nabla_{\alpha'(t)}Y(t), Y(t)\rangle = 0$

$\frac{d}{dt}\langle X(t), Y(t)\rangle = \langle \nabla_{\alpha'(t)}X(t), Y(t)\rangle + \langle X(t), \nabla_{\alpha'(t)}Y(t)\rangle = 0$